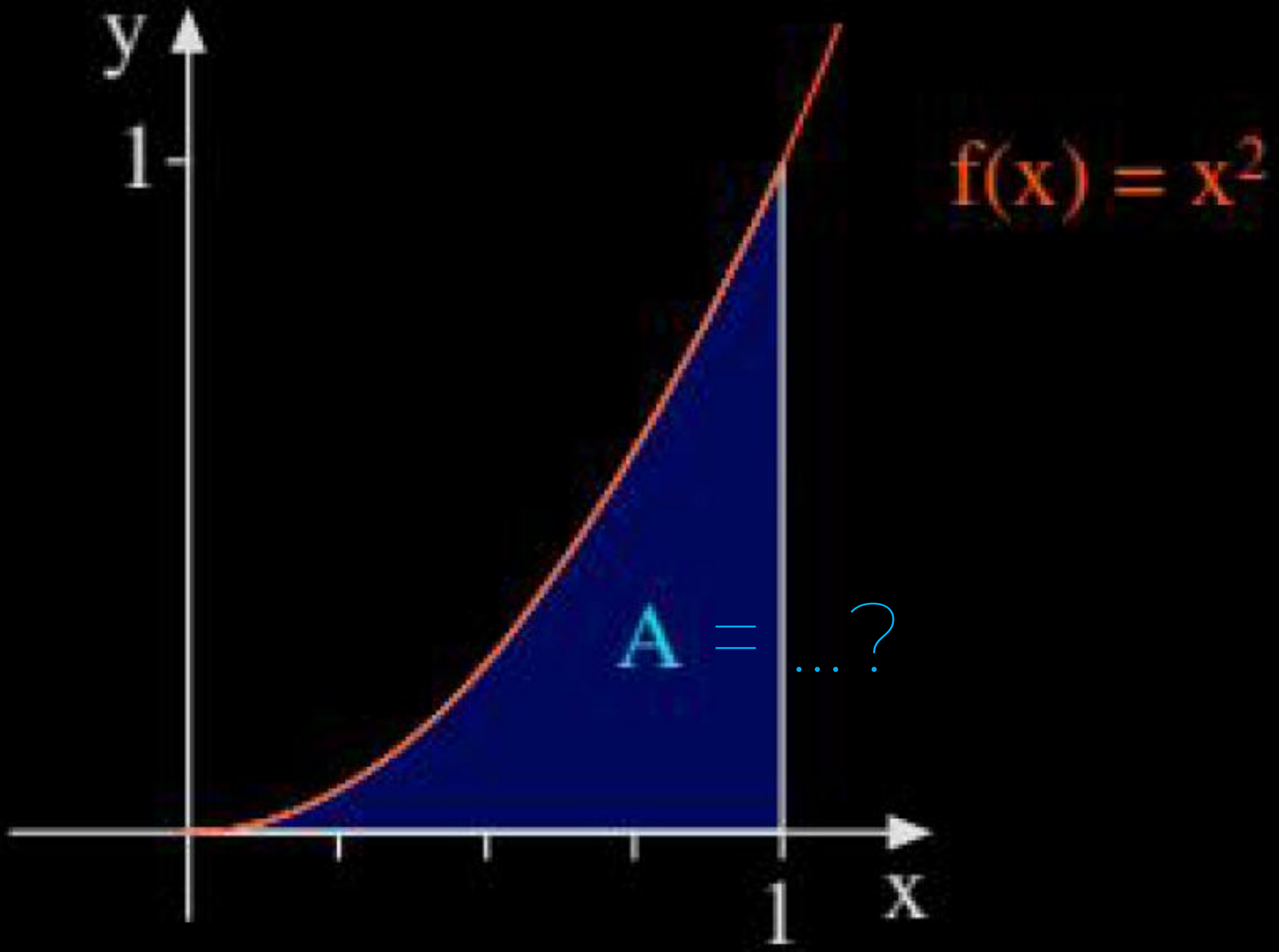
Mathematatik Q1 Abels

Kopfübung

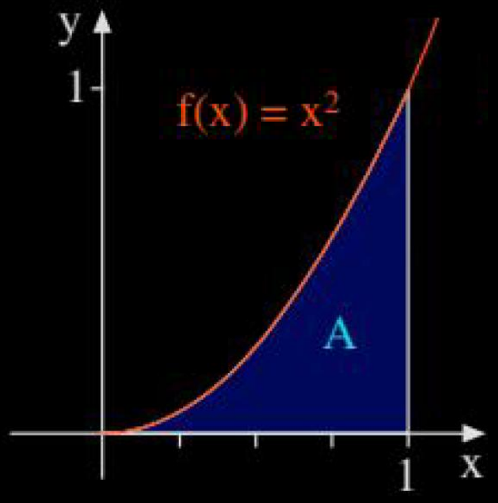
- $\frac{1}{4} \cdot \left(\frac{0}{4}\right)^2 + \frac{1}{4} \cdot \left(\frac{1}{4}\right)^2 + \frac{1}{4} \cdot \left(\frac{2}{4}\right)^2 + \frac{1}{4} \cdot \left(\frac{3}{4}\right)^2 = \cdots$

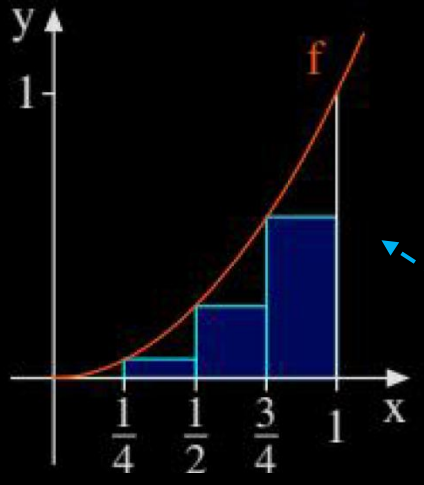
- $\frac{1}{4} \cdot \left(\frac{1}{4}\right)^2 + \frac{1}{4} \cdot \left(\frac{2}{4}\right)^2 + \frac{1}{4} \cdot \left(\frac{3}{4}\right)^2 + \frac{1}{4} \cdot \left(\frac{4}{4}\right)^2 = \cdots$

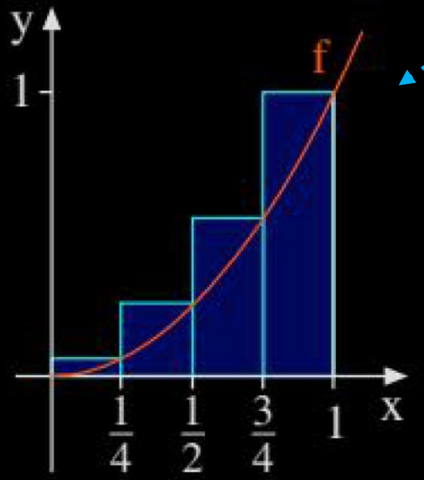
Die Streifenmethode des Archimedes


y
1
f(x) = x²
A = ...?
1
x

Streifenmethode: Einschachtelung







Untersumme $U_4 \leq A \leq$ Obersumme O_4

$$U_4 = \frac{1}{4} \cdot \left[0^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{2}{4}\right)^2 + \left(\frac{3}{4}\right)^2\right] = \frac{14}{64}$$

$$O_4 = \frac{1}{4} \cdot \left[\left(\frac{1}{4}\right)^2 + \left(\frac{2}{4}\right)^2 + \left(\frac{3}{4}\right)^2 + 1^2\right] = \frac{30}{64}$$

$$\frac{14}{64} \leq A \leq \frac{30}{64}$$

$$0{,}21 \leq A \leq 0{,}47$$

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3. Berechnen Sie U_4 und O_4 sowie U_8 und O_8 für die angegebene Funktion f über dem Intervall I.

a) $f(x) = x + 1$, $I = [0; 1]$

b) $f(x) = 2 - x$, $I = [0; 2]$

c) $f(x) = \frac{1}{2}x^2$, $I = [0; 1]$

d) $f(x) = x^2$, $I = [1; 2]$

e) $f(x) = 2x^2 + 1$, $I = [0; 2]$

f) $f(x) = x^4$, $I = [0; 2]$

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3.

a) $U_4 = \frac{11}{8} = 1.375, O_4 = \frac{13}{8} = 1.625, U_8 = \frac{23}{16} = 1.4375, O_8 = \frac{25}{16} = 1.5625$

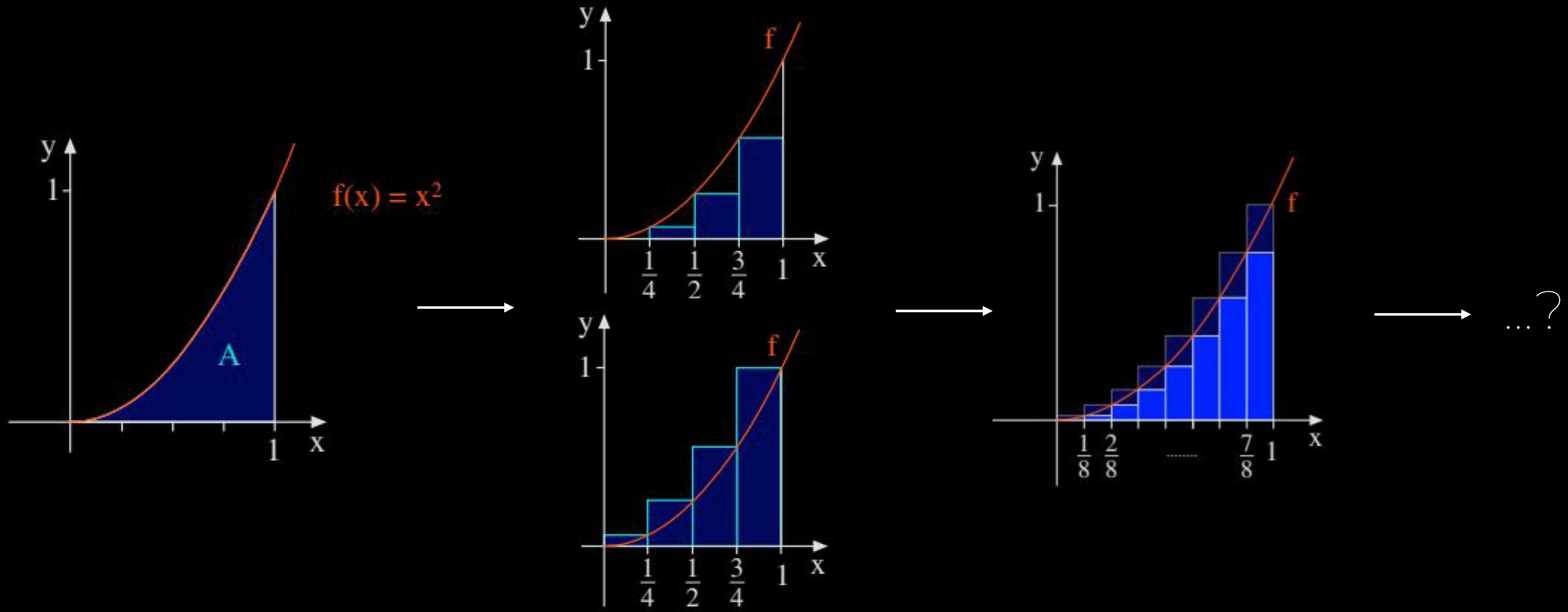
b) $U_4 = \frac{3}{2} = 1.5, O_4 = \frac{5}{2} = 2.5, U_8 = \frac{7}{4} = 1.75, O_8 = \frac{9}{4} = 2.25$

c) $U_4 = \frac{7}{64} = 0.109375, O_4 = \frac{15}{64} = 0.234375, U_8 = \frac{35}{256} \approx 0.14, O_8 = \frac{51}{256} \approx 0.2$

d) $U_4 = \frac{63}{32} = 1.96875, O_4 = \frac{87}{32} = 2.71875, U_8 = \frac{275}{128} \approx 2.15, O_8 = \frac{323}{128} \approx 2.52$

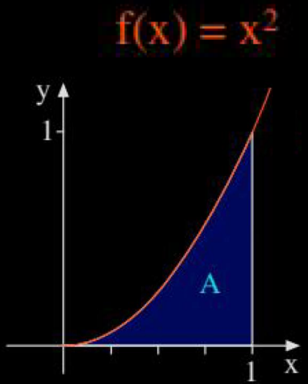
e) $U_4 = \frac{11}{2} = 5.5, O_4 = \frac{19}{2} = 9.5, U_8 = \frac{51}{8} = 6.375, O_8 = \frac{67}{8} = 8.375$

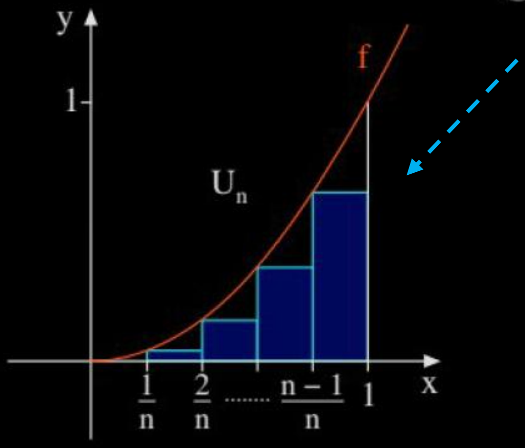
f) $U_4 = \frac{49}{16} = 3.0625, O_4 = \frac{177}{16} = 11.0625, U_8 = \frac{1169}{256} \approx 4.56, O_8 = \frac{2193}{256} \approx 8.57$



y
1
f(x) = x²
A
1
x
y
1
f
1/4
1/2
3/4
1
x
y
1
f
1/4
1/2
3/4
1
x
y
1
f
1/8
2/8
7/8
1
x
...?

Streifenmethode: Exakt





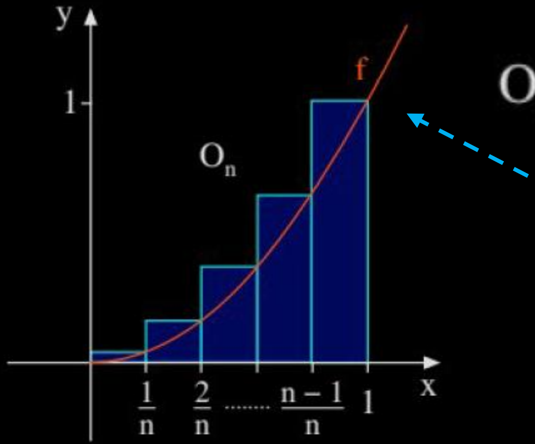
$$U_n = \frac{1}{n} \cdot \left[0^2 + \left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \ldots + \left(\frac{n-1}{n}\right)^2\right]$$

$$= \frac{1}{n} \cdot \left[0^2 + \frac{1^2}{n^2} + \frac{2^2}{n^2} + \ldots + \frac{(n-1)^2}{n^2}\right]$$

$$= \frac{1}{n^3} \cdot \left[0^2 + 1^2 + 2^2 + \ldots + (n-1)^2\right]$$

$$= \frac{1}{n^3} \cdot \frac{1}{6} \cdot (n-1) \cdot n \cdot (2n-1)$$

$$= \frac{1}{6} \cdot \frac{n-1}{n} \cdot \frac{n}{n} \cdot \frac{2n-1}{n} \xrightarrow[n \to \infty]{} \frac{1}{6} \cdot 1 \cdot 1 \cdot 2 = \frac{1}{3}$$



$$O_n = \frac{1}{n} \cdot \left[\left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \ldots + \left(\frac{n-1}{n}\right)^2 + 1^2\right]$$

$$= \frac{1}{n^3} \cdot \left[1^2 + 2^2 + \ldots + (n-1)^2 + n^2\right]$$

$$= \frac{1}{n^3} \cdot \frac{1}{6} \cdot n \cdot (n+1) \cdot (2n+1)$$

$$= \frac{1}{6} \cdot \frac{n}{n} \cdot \frac{n+1}{n} \cdot \frac{2n+1}{n} \xrightarrow[n \to \infty]{} \frac{1}{6} \cdot 1 \cdot 1 \cdot 2 = \frac{1}{3}$$

$$\lim_{n \to \infty} U_n \leq A \leq \lim_{n \to \infty} O_n$$

$$\frac{1}{3} \leq A \leq \frac{1}{3}$$

$$A = \frac{1}{3}$$

!

Big17

4. Berechnen Sie U_n und O_n für die Funktion f über dem Intervall I. Welcher Grenzwert ergibt sich jeweils für $n \to \infty$?

a) $f(x) = x + 1, \quad I = [0; 1]$

b) $f(x) = 2 - x, \quad I = [0; 2]$

c) $f(x) = x^2, \quad I = [0; 10]$

d) $f(x) = 2x^2 + x, \quad I = [0; 1]$

Benötigte Summenformeln: $1 + 2 + \ldots + n = \frac{n(n+1)}{2}$, $\quad 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

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4.

a) $U_n = \frac{3n-1}{2n} \xrightarrow[n\to\infty]{} \frac{3}{2}, O_n = \frac{3n+1}{2n} \xrightarrow[n\to\infty]{} \frac{3}{2}$

b) $U_n = 2 - \frac{2}{n} \xrightarrow[n\to\infty]{} 2, O_n = 2 + \frac{2}{n} \xrightarrow[n\to\infty]{} 2$

c) $U_n = \frac{500}{3} \cdot \frac{(n-1)(2n-1)}{n^2} \xrightarrow[n\to\infty]{} \frac{1000}{3}, O_n = \frac{500}{3} \cdot \frac{(n+1)(2n+1)}{n^2} \xrightarrow[n\to\infty]{} \frac{1000}{3}$

d) $U_n = \frac{(n-1)(2n-1)}{3n^2} + \frac{n-1}{2n} \xrightarrow[n\to\infty]{} \frac{7}{6}, O_n = \frac{(n+1)(2n+1)}{3n^2} + \frac{n+1}{2n} \xrightarrow[n\to\infty]{} \frac{7}{6}$

5. Gesucht ist der Inhalt A der Fläche zwischen dem Graphen von $f(x) = x^3$ und der x-Achse über dem Intervall [0; 1].
Gehen Sie analog zum Beispiel $f(x) = x^2$ (S. 16) vor.

Benötigte Summenformel: $1^3 + 2^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}$